

Incompressible Surfaces in
Hyperbolic Four-Punctured Sphere Bundles

by

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This dissertation by Yang Xiao is accepted in its present form by
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For my mom, without whom nothing is possible

献给我的母亲

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Prologue

Mathematicians seem to have access to a different world from the mundane, earthly one that I grew up in. It is a world that has intrigued me, enchanted me, and inspired me. Seemingly different pieces always find surprising connections to each other and all hints point to an invisible set of rules that seem to govern the mathematical universe with great precision and subtlety. At the same time though, it is a world that has intimidated me, isolated me, and made me constantly feel inadequate. I almost burst into tears when I read Simone Weil's confession in Francis Su's *Mathematics for Human Flourishing*: "I did not mind having no visible successes, what did grieve me was the idea of being excluded from that transcendent kingdom to which only the truly great have access and wherein truth abides." Simone wrote this standing in the shadow of her brother, the renowned mathematician André Weil, feeling small and rejected. I have many times felt the same sitting in front of the mathematical giants in my field, sensing a vast and deep universe gracing me with its presence but only offering its keys to a selected few.

My mind flashed back to a visit to the most renowned university in China with my mom nearly ten years ago. I took her there to see a movie at the theater on campus, wanting to show her the impressive institution that many of my high school classmates were studying at, one that I had also been admitted to but ultimately declined. She was ginger with her steps when we sauntered around after the movie, her eyes full of excitement and admiration, yet tainted with an almost undetectable sense of longing. "It is so beautiful, so beautiful here." she mumbled.

The scene struck me now with its vividness, but even more with its heaviness now that I have a deepened understanding of the complex mix of emotions she must have experienced that night. My mom has never gone to college. None of my family did. She had to drop out of middle school to support herself and her younger brother. Thirty years later, she had created a completely different life for me, one where I got to decline a prestigious university at home to pursue a dream 6000 miles away, simply to open up my eyes and experience different worlds. I imagine what it must have felt like for her to step into that campus, a sacred world of knowledge and truths that she had fought so hard for her daughter to enter but never got the key herself.

There are so many doors between the worlds that separate people on this earth, and I have been on both sides of those doors on so many occasions. Some doors opened up to me without me realizing, and as a consequence I enjoyed the blissful entitlement to my health, my safety and security, my access to education, my happiness, and the love and support of my family and friends. Nevertheless it is those doors that stubbornly rejected me that taught me how privileged I have been, and how unjust it is to deny anyone the access to the same wholeness I was able to pursue, such as the unique and sublime experience of studying mathematics.

I fell in love with math when I took my first topology class in junior year of college. I was struck by the wealth of imagination, the elegance, and the regularity of the mathematical world. A brand new infinite dimensional universe opened up to me and I have aspired to become a topologist ever since. But math is hard. I crawled underneath my desk and bawled for a whole hour after I nearly failed my first analysis exam. I had never felt that thwarted in any of my endeavors before. Then things kept getting harder. The first time I attempted to read Thurston's Hyperbolic structures on 3-manifolds II, I stared at the same paragraph for hours and even days without being able to develop any mental picture of what was going on. The same feeling persisted through the rest of my PhD, including when I first picked up the paper that inspired this thesis. I cannot remember how many times I thought to myself that I would not be able to do this, that I was not smart enough

to do research, that I had been banging on a door that would never open, that I should quit.

But I didn't. Despite all these difficult moments, two things have kept me going. The first is the realization that persistence helps. Even though one is not guaranteed to find any answer simply by staring at the problem long enough, I find that my overall mathematical understanding has deepened over time so long as I do not give up trying. In the last few years I have revisited the papers that inspired my thesis project a dozen of times, and every time I managed to learn something new or found some clarity that led to new ideas. As long and as uncertain as the process is, nothing beats the joy of finding truth beneath all the mysteries and chaos. In those few light bulb moments where structure and order finally revealed themselves to me, my frustration went away instantly and every struggle seemed to be worth it - the joy was simply pure and transcendent.

The second, which is even more crucial, is the support and companionship from my close friends in the mathematical community. The door to the mathematical world is a stubborn one, and a proud one. Many people, from either side of that door, believe that the access belongs only to "the truly great" and that true greatness is an innate quality. Many of them made sure to let me know that I am deemed unworthy for such access: from my sixth grade teacher who thought girls excelling in math would always be caught up by boys by middle school because "girls don't have the math genes", to one of the most prominent mathematicians in my field who stopped responding to my emails after I asked a "very basic question", many mathematical figures in my life have made me doubt my fitness for pursuing mathematical research. Yet fortunately for me, I met friends and mentors who silenced every voice of doubt for me with their encouragement and acceptance. What they did was more than calming my nerves with kind words. They took my hand and invited me to work on research projects together. They treated me with attention and respect, and believed in my ability to contribute equally. I have never experienced more fun and felt more sense of belonging in mathematics than in collaborations, where not surprisingly, I have also been the most productive. Their support not only carried me through my darkest

moments, but also gave me the courage to continue on in this PhD journey.

When I started this thesis project, I imagined it would become my ticket to the mathematical world and my induction into the mathematicians' honor society. I now see it completely differently. This thesis is a gift from myself, the young girl who has grown up so much in the past six years, to the future me as a reminder of how essential perseverance and community are in the process of dismantling doors built on privileges, misconceptions, and stereotypes. It is a gift that I want to dedicate to my mom, who has taught me the same lesson long ago through her own mighty resistance to injustice and misfortune. Most of all, it is a gift that I want to share with anyone who has felt the same way as Simone did in front of that invisible door:

If the door rejects you, then let's take it down.